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TIME-HARMONIC BODY FORCE LOADING OF A MODE-I PENNY-SHAPED CRACK

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Abstract The present paper is concerned with the problem of determining dynamic SIF of a penny-shaped crack in an infinite elastic medium, which is subjected to the action of time-harmonic axial body forces, placed symmetrically with respect to the crack plane. The solution of the problem is obtained by superposition of the solutions of two simpler problems. The first of these problems is related to the *unperturbed (crackless)* space under the prescribed axial body forces, while the second problem consists in finding the dynamic SIF of the penny-shaped crack whose faces are *directly* acted upon by some axial stresses. The form of these axial stresses is determined from the solution of the first problem. Fourier and Hankel transforms have been used to solve the first problem. Next by means of the Hankel transform, the second problem has been reduced to a pair of dual integral equations which have been subsequently transformed into a Fredholm integral equation of the second kind via an auxiliary function. The integral equation has been solved numerically in order to determine the variations of the dynamic stress intensity factor at the rim of the penny-shaped crack for some particular body force loading cases.

1. INTRODUCTION

Problems of cracks and inclusions under dynamic loadings continue to attract attention of the researchers because of their numerous practical applications, especially in seismology, non-destructive evaluation and geophysics. However in this area the bulk of the previous investigations are restricted to the consideration of the cases where the elastic bodies containing cracks are subjected to the action of loads placed directly on the crack borders or the elastic waves impinging on the crack surfaces originate from a source situated at infinity. From the mathematical point of view, these cases are identical. However in many practical situations, one frequently encounters problems of determining stress intensity factors for cracks or inclusions in bodies supported or strengthened by stiffeners, stringers, rivets etc. In practical considerations, the reactions of these strengthening materials are replaced by body forces. In elastostatics, starting with Sneddon and Tweed's (1967a, b, 1971) and Tweed's works (1969a, b), there appeared quite a good number of papers relating to mixed boundary value problems of elasticity, involving body forces. Excellent reviews of works in this direction can be found in Kassir and Sih (1975), Galin (1976), Andreikiv (1982) and Fabrikant (1991). However in elastodynamics such solutions are scarce. Probably it was Borodachev (1974) who first treated an elastodynamic crack problem involving body forces.

The present paper is concerned with the axisymmetric problem of determining the dynamic stress intensity factor of a penny-shaped crack in an infinite elastic solid in which time–harmonic axial body forces are available. Solution of the title problem has been obtained by superposition of the solutions of two simpler problems. The first problem is concerned with the determination of the elastodynamic field in an *unperturbed or crackless* space under the prescribed body forces, while the second problem consists of finding the dynamic stress intensity factor of the penny-shaped crack whose faces are *directly* acted upon by some axial stresses. The form of these stresses is determined from the solution of the first problem. Fourier and Hankel transforms have been employed to solve the first problem. Essentially the solution of the first problem explores the elastodynamic field induced in an infinite space by the action of symmetrically placed time-harmonic axial body

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forces. Next Hankel transform has been used to reduce the second problem to a pair of dual integral equations which have been subsequently transformed into a Fredholm integral equation of the second kind via an auxiliary function. The integral equation has been solved numerically in order to determine the variations of the dynamic SIF at the vicinity of the penny-shaped crack with the longitudinal wave number for different values of the body force placement distance for two body force loading cases, of which the first corresponds to the case where the elastic body with the penny-shaped crack is stretched by symmetrically placed uniform axial loads of constant intensity, harmonically changing in time, whilst the second one corresponds to the case where the elastic body is subjected to the action of symmetrically placed concentrated loads uniformly distributed over a circular region. The problem of determining the dynamic SIF of a penny-shaped crack in an elastic solid under time–harmonic *torsional* body forces has also been treated in a similar fashion in a recent work by the author (Rahman, 1994).

In fact, the present paper may be regarded as an extension of the well-known works by Robertson (1967) and Sih and Loeber (1969). Problems of diffraction of elastic waves at a penny-shaped crack have also been investigated by Mal (1968), Achenbach et al (1978), Srivastava *et al* (1982), Krenk and Schimdt (1982), Martin and Wickham (1983), Keogh (1986), Lin and Keer (1987), Budreck and Achenbach (1988) and others. Good reviews of works in this direction can be found in Kraut (1976), Sih (1977), Guz *et al* (1978), Slepyan (1981), Parton and Boriskovsky (1990) and Freund (1990).

2. FORMULATION OF THE PROBLEM

Consider a cylindrical coordinate system (r, θ, z) with the origin at the center of a penny-shaped crack and let the crack occupy the region $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$, $z = 0$. The crack is assumed to be excited by the stress waves caused by the action of axially symmetric time-harmonic axial body forces $F_z(r, z, t) = F_z^*(r, z)\exp(i\omega t)$ placed symmetrically with respect to the crack plane $z = 0$ at a finite distance from it.

The title problem reduces to that of finding the solution of the following partial differential equations of the motion of the elastic medium (Lurie, 1970):

$$\mu\left(\nabla u_r - \frac{u_r}{r^2}\right) + (\lambda+\mu)\frac{\partial e}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}$$

$$\mu\nabla u_z + (\lambda+\mu)\frac{\partial e}{\partial z} + F_z = \rho\frac{\partial^2 u_z}{\partial t^2} \tag{1}$$

where λ, μ are the Lamé's constants, ρ is the density of the material of the elastic medium, u_r and u_z are the non-zero components of the displacement vector, $e = (1/r)\partial(ru_r)/\partial r + \partial u_z/\partial z$ is the dilatation and $\nabla = \partial^2/\partial r^2 + (1/r)\partial/\partial r + \partial^2/\partial z^2$ is the axisymmetric Laplacian.

The non-zero components of the stress tensor are related to the those of the displacement vector by the following relations:

$$\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} + \lambda e$$

$$\sigma_{zz} = 2\mu\frac{\partial u_z}{\partial z} + \lambda e$$

$$\sigma_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$\sigma_{\theta\theta} = 2\mu\frac{u_r}{r} + \lambda e. \tag{2}$$

The solution of the equations (1) is subject to the following mixed boundary conditions:

$$\sigma_z(r,0,t) = 0, 0 \leqslant r < a$$
$$u_z(r,0,t) = 0, r > a$$
$$\sigma_{rz}(r,0,t) = 0, 0 \leqslant r < \infty \tag{3}$$

In the sequel, we shall consider time-harmonic vibration of the elastic space, which means that all field quantities vary in time as $\exp(i\omega t)$, where ω is the circular frequency of vibration and $i = \sqrt{-1}$ is the imaginary unit. In view of this, the elastodynamic equations (1) take the following form:

$$\nabla u_r^* - \frac{u_r^*}{r^2} + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{\partial e^*}{\partial r} + k_2^2 u_r^* = 0$$
$$\nabla u_z^* + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{\partial e^*}{\partial z} + k_2^2 u_z^* = -\frac{F_z^*}{\mu} \tag{4}$$

where $\varepsilon = c_2/c_1 = [(1-2\nu)/2(1-\nu)]^{1/2}$ (ν is the Poisson's ratio of the material of the elastic medium), $k_i = \omega/c_i (i = 1,2)$ and $c_1 = \sqrt{(\lambda+2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ are the dilatational and shear wave speeds, respectively. Here the asterisked quantities denote the complex amplitudes of the corresponding field quantities.

In the subsequent analysis, the time-factor $\exp(i\omega t)$, common to all field quantities, will be omitted but understood.

Within the framework of linear elasticity, the solution of the formulated problem can be obtained by superposition of the solutions of two simpler problems. They are as follows:

Problem 1.

It is required to solve the equations

$$\nabla u_r^* - \frac{u_r^*}{r^2} + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{\partial e^*}{\partial r} + k_2^2 u_r^* = 0$$
$$\nabla u_z^* + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{\partial e^*}{\partial z} + k_2^2 u_z^* = -\frac{F_z^*}{\mu} \tag{5}$$

with the boundary conditions

$$u_z^*(r,z) = 0, 0 \leqslant r < \infty$$
$$\sigma_{rz}^*(r,0) = 0, 0 \leqslant r < \infty \tag{6}$$

In fact this problem consists in determining the elastic field in an *unperturbed or crackless* space under the action of the prescribed body forces $F_z(r,z,t)$. Obviously this problem is a *non-mixed* one. For this problem it is an easy matter to find the normal stress $\sigma_z^{*(1)}(r,0)$ (the superscript 1 denotes the quantity pertaining to the first problem).

Problem 2 (*the perturbation problem*).

To find the solution of the following equations

$$\nabla u_r^* - \frac{u_r^*}{r^2} + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{\partial e^*}{\partial r} + k_2^2 u_r^* = 0$$
$$\nabla u_z^* + \left(\frac{1}{\varepsilon^2} - 1\right)\frac{\partial e^*}{\partial z} + k_2^2 u_z^* = 0 \tag{7}$$

subject to the *mixed* boundary conditions

$$\sigma_z^*(r,0) = -p(r), 0 \leqslant r < a$$
$$u_z^*(r,0) = 0, r > a$$
$$\sigma_{rz}^*(r,0) = 0, 0 \leqslant r < \infty \tag{8}$$

where $p(r) = \sigma_z^{*(1)}(r,0)$.

In other words, we have to find the stress distribution in the elastic space with the penny-shaped crack whose faces are acted upon by some dynamic normal stresses $\sigma_z^{*(1)}(r,0)\exp(i\omega t)$.

Solutions of both problems must also satisfy the radiation conditions so that the outgoing waves have null fields at infinity.

3. SOLUTION OF THE FIRST PROBLEM

In order to solve the first problem, we introduce the following mixed Fourier–Hankel transforms (Sneddon, 1972):

$$\mathscr{F}\{\mathscr{H}_1[u_r^*(r,z);\quad r\to s];\quad z\to\alpha\} = U(s,\alpha)$$
$$\mathscr{F}\{\mathscr{H}_0[u_z^*(r,z);\quad r\to s];\quad z\to\alpha\} = V(s,\alpha)$$
$$\mathscr{F}\{\mathscr{H}_0[F_z^*(r,z);\quad r\to s];\quad z\to\alpha\} = \Omega(s,\alpha) \tag{9}$$

where

$$\mathscr{F}\{f(r,z);\quad z\to\alpha\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(r,z)\exp(i\alpha z)\,dz$$

$$\mathscr{H}_\nu[f(r,z);\quad r\to s] = \int_0^\infty rf(r,z)J_\nu(rs)\,dr$$

are the Fourier transform and Hankel transform of the νth order, respectively of the function $f(r,z)$, and $J_\nu(rs)$ is the Bessel function of the first kind and of the νth order.

Application of these transformations to the equations (4) and subsequent inversions of the Fourier and Hankel transforms, lead to the following expressions for the components of the displacement vector:

$$u_r^*(r,z) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty sJ_1(rs)\,ds\int_{-\infty}^{\infty}\psi_1(s,\alpha)\Omega(s,\alpha)\exp(-i\alpha z)\,d\alpha$$
$$u_z^*(r,z) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty sJ_0(rs)\,ds\int_{-\infty}^{\infty}\psi_2(s,\alpha)\Omega(s,\alpha)\exp(-i\alpha z)\,d\alpha \tag{10}$$

where

$$\psi_1(s,\alpha) = \frac{is\alpha(1-\varepsilon^2)}{\mu(s^2-k_1^2+\alpha^2)(s^2-k_2^2+\alpha^2)}$$
$$\psi_2(s,\alpha) = \frac{s^2-k_1^2+\varepsilon^2\alpha^2}{\mu(s^2-k_1^2+\alpha^2)(s^2-k_2^2+\alpha^2)}. \tag{11}$$

Note that the functions $\psi_1(s,\alpha)$ and $\psi_2(s,\alpha)$ are odd and even, respectively, with respect to α.

Since we are considering body forces that are *symmetric with respect to the plane* $z = 0$, the function $F_z^*(r,z)$ is *odd* in z and hence the following relation holds:

$$\Omega(s,\alpha) = i\Omega_s(s,\alpha)$$

where

$$\Omega_s(s,\alpha) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty rJ_0(rs)\,dr\int_0^\infty F_z^*(r,z)\sin(\alpha z)\,dz$$

so that the function $\Omega_s(s,\alpha)$ is odd in α. Considering this and also the fact that the functions ψ_1 and ψ_2 are respectively odd and even with respect to α, we replace the Fourier exponential inversions in eqn (10) by Fourier cosine and sine inversions, respectively, thus obtaining

$$u_r^*(r,z) = i\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty sJ_1(rs)\,ds\int_0^\infty \psi_1\Omega_s\cos(\alpha z)\,d\alpha$$

$$u_z^*(r,z) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty sJ_0(rs)\,ds\int_0^\infty \psi_2\Omega_s\sin(\alpha z)\,d\alpha \tag{12}$$

Let us now turn to the determination of the stress field. Putting eqn (10) into the expressions (2_2) and (2_3), after some simple manipulations, we obtain

$$\sigma_z^*(r,z) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty rJ_0(rs)\,dr\int_{-\infty}^\infty \Omega(s,\alpha)\Omega_\sigma(s,\alpha)\exp(-i\alpha z)\,d\alpha$$

$$\sigma_{rz}^*(r,z) = \mu\left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty rJ_1(rs)\,dr\int_{-\infty}^\infty (i\alpha\psi_1+s\psi_2)\Omega(s,\alpha)\exp(-i\alpha z)\,d\alpha \tag{13}$$

where

$$\Omega_\sigma(s,\alpha) = \frac{i\alpha(k_2^2-\alpha^2-3s^2+2\varepsilon^2s^2)}{(s^2-k_1^2+\alpha^2)(s^2-k_2^2+\alpha^2)}. \tag{14}$$

Again, considering the oddness and evenness of the functions ψ_1 and ψ_2, we can rewrite the expression for $\sigma_{rz}^*(r,z)$ as

$$\sigma_{rz}^*(r,z) = \mu\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty rJ_1(rs)\,dr\int_0^\infty (i\alpha\psi_1+s\psi_2)\Omega_s(s,\alpha)\sin(\alpha z)\,d\alpha. \tag{15}$$

Putting $z = 0$ into the expressions for $u_z^*(r,z)$ in eqn (12) and $\sigma_{rz}^*(r,z)$ in eqn (15), it can be easily seen that the boundary conditions of the first problem given by eqn (6) are completely satisfied. Since the solution of the second problem necessitates determination of the function $\sigma_z^{*(1)}(r,0)$, in the sequel we shall focus our attention exclusively on the function $\sigma_z^*(r,z)$. Our next step is to transform the expression for $\sigma_z^*(r,z)$ in eqn (13) into a more suitable form, since the solution (13_1) is given in terms of the Fourier and Hankel transforms.

Thus using Faltung theorem (Sneddon, 1972) for the Fourier transform in the expression for $\sigma_z^*(r,z)$ in eqn (13), we obtain

$$\sigma_z^*(r,z) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty sJ_0(rs)\,ds\int_{-l}^{l}\tilde{\Omega}_\sigma(s,z-z_0)\tilde{\Omega}(s,z_0)\,dz_0 \tag{16}$$

where

$$\tilde{\Omega}_\sigma(s,z) = \mathscr{F}^{-1}\{\Omega_\sigma(s,\alpha)\,;\alpha \to z\}$$
$$\tilde{\Omega}(s,z) = \mathscr{F}^{-1}\{\Omega(s,\alpha)\,;\alpha \to z\}. \tag{17}$$

Here $\mathscr{F}^{-1}$ denotes Fourier inverse transform.

Note that the function $\tilde{\Omega}(s,z_0)$, being the inverse Fourier transform of the function $\Omega(s,\alpha)$, is the Hankel transform of the function $F_z^*(r,z_0)$ and hence we can write

$$\tilde{\Omega}(s,z_0) = \int_0^\infty r_0 F_z^*(r_0,z_0) J_0(r_0 s)\,\mathrm{d}r_0. \tag{18}$$

Putting eqn (18) into eqn (16) and changing the order of integration, we obtain

$$\sigma_z^*(r,z) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty r_0\,\mathrm{d}r_0 \int_0^\infty sJ_0(rs)J_0(r_0 s)\,\mathrm{d}s \int_{-\infty}^{\infty} \tilde{\Omega}_\sigma(s,z-z_0)F_z^*(r_0,z_0)\,\mathrm{d}z_0 \tag{19}$$

For the product of the Bessel functions in eqn (19), we use the following expression due to Neumann (Webster, 1927):

$$J_0(rs)J_0(r_0 s) = \frac{1}{\pi}\int_0^\pi J_0(r's)\,\mathrm{d}\phi \tag{20}$$

where $r' = (r^2+r_0^2-2rr_0\cos\phi)^{1/2}$.

Insertion of eqn (20) into eqn (19) leads to the following expression for $\sigma_z^*(r,z)$:

$$\sigma_z^*(r,z) = 2\pi \int_0^\infty r_0\,\mathrm{d}r_0 \int_{-\infty}^{\infty} G_\sigma^*(r,r_0\,;z,z_0)F_z^*(r_0,z_0)\,\mathrm{d}z_0 \tag{21}$$

where

$$G_\sigma^*(r,r_0\,;z,z_0) = \frac{1}{\pi(2\pi)^{3/2}}\int_0^\pi \Psi^*(r',z-z_0)\,\mathrm{d}\phi \tag{22}$$

Here

$$\Psi^*(r,z) = \mathscr{H}_0[\mathscr{F}^{-1}\{\Omega_\sigma(s,\alpha)\,;\alpha\to z\}\,;s\to r] \tag{23}$$

Performing Fourier and Hankel inversions in eqn (23), the following closed-form expression has been found for $\Psi^*(r,z)$:

$$\Psi^*(r,z) = -\left(\frac{\pi}{2}\right)^{1/2} z\Bigg[(1-2\varepsilon^2)\frac{iR_0k_1+1}{R_0^3}\exp(-ik_1R_0) + 2\frac{ik_2R_0+1}{R_0^3}\exp(-ik_2R_0) + \frac{2}{k_2^2}\sum_{j=1}^{2}(-1)^j\Bigg\{\frac{3(R_0^2k_j^2-3iR_0k_j-3)}{R_0^5} \times\exp(-ik_jR_0) + \frac{15-iR_0^3k_j^3+15iR_0k_j-6R_0^2k_j^2}{R_0^7}z^2\exp(-ik_jR_0)\Bigg\}\Bigg] \tag{24}$$

where $R_0 = (r^2+z^2)^{1/2}$.

Note that the function $\Psi^*(r,z)$ is odd in z.

The structure of the eqn (21) clearly shows that $G_{\sigma}^{*}(r, r_0; z, z_0)$ represents the required Green's function of the first problem, which corresponds to the action of a unit time-harmonic concentrated load acting in the axial direction and uniformly distributed along a circular ring of radius r_0 at a distance z_0 from the z-plane. The correctness of the solutions (21), (22) and (24) has been established in a recent paper by the author (Rahman, 1995).

The expressions for the displacement components given by eqn (10) can also be transformed into more suitable forms as we did in case of the function $\sigma_z^{*}(r, z)$. However in the present analysis we omitted these transformations, insofar as in the sequel we will only need the expression for $\sigma_z^{*}(r, 0)$.

It is also worth mentioning in this context that the problem of determining elastodynamic fields in an infinite space under the action of time-varying body forces was also considered by Eason *et al* (1956). However the solution was given in the domain of the corresponding integral transforms. Evidently such a solution is of limited importance, since in each body force loading case, one is to find the corresponding transforms of the body force loading function and evaluate a number of integrals. In this sense the present approach is *direct*.

Expressions (21), (22) and (24) give required solutions to the first problem. However in order to solve the second problem, we need expressions for $\sigma_z^{*}(r, z)$ for the case where $z = 0$. Thus putting $z = 0$ into eqn (21), we get

$$\sigma_z^{*}(r, 0) = 2\pi \int_0^{\infty} r_0 \, \mathrm{d}r_0 \int_{-\infty}^{\infty} G_{\sigma}^{*}(r, r_0; 0, z_0) F_z^{*}(r_0, z_0) \, \mathrm{d}z_0 \tag{25}$$

The expressions (22), (24) and (25) are sufficient to turn to the second problem. However before proceeding to solve the second problem, let us consider some cases of the body force loadings of the elastic space.

Example 1: let the elastic space be subjected to two concentrated loads $P\exp(i\omega t)$, acting in the axial direction, placed symmetrically with respect to the plane $z = 0$ at a distance z_1 from it. In this case, we have

$$F_z^{*}(r, z) = \frac{P}{2\pi r} \delta(r)[\delta(z - z_1) - \delta(z + z_1)] \tag{26}$$

where $\delta(\ldots)$ is the dirac's delta function.

Putting eqn (26) into eqn (25) and considering that the function $\Psi^{*}(r, z)$ is odd with respect to z, we get

$$\sigma_z^{*}(r, 0) = \frac{2P}{(2\pi)^{3/2}} \Psi^{*}(r, z). \tag{27}$$

The corresponding static solution can be obtained by letting ω tend to zero in eqns (27) and (24). Omitting intermediate calculations, we record the final result as

$$\sigma_z(r, 0) = \frac{P z_1}{4\pi(1 - \nu)} \left(\frac{1 - 2\nu}{R^3} + 3\frac{z_1^2}{R^5} \right) \tag{28}$$

The expression (28) is accurately the same as that given in Lurie (1970).

Example 2: consider the case where the elastic space is subjected to the action of uniformly distributed time-harmonic axial loads of constant intensity σ over a circular region of radius c, placed symmetrically with respect to the plane $z = 0$ at a distance $z = z_1$ from it. In this case,

$$F_z^*(r,z) = \sigma H(c-r)[\delta(z-z_1)-\delta(z+z_1)] \tag{29}$$

where $H(\ldots)$ is the Heaviside step function.

Putting eqn (29) into eqn (25), we obtain

$$\sigma_z^*(r,0) = 4\pi\sigma \int_0^c r_0 G_\sigma^*(r,r_0;0,z_1)\,dr_0 \tag{30}$$

Analogous expressions can be derived for any axisymmetric time-harmonic body force loading cases.

We now turn to the solution of the second problem.

4. SOLUTION OF THE SECOND PROBLEM (THE PERTURBATION PROBLEM)

In view of the symmetry of the second problem with respect to the plane $z = 0$, it is sufficient to consider only one half-space, say, the half-space, $z \geqslant 0$.

Now applying Hankel transform, the solution of the equations (7), satisfying radiation conditions, can be represented as

$$\begin{aligned} u_r^*(r,z) &= \mathscr{H}_1[A_1\exp(-\gamma_1 z)+A_2\exp(-\gamma_2 z);s\to r] \\ u_z^*(r,z) &= \mathscr{H}_0[\gamma_1 s^{-1}A_1\exp(-\gamma_1 z)+s\gamma_2^{-1}A_2\exp(-\gamma_2 z);s\to r] \end{aligned} \tag{31}$$

where A_j $(j = 1, 2)$ are the unknown coefficients to be determined using the boundary conditions (8) and $\gamma_j^2 = s^2-k_j^2$, $(j = 1, 2)$. As per the radiation conditions at the time factor $\exp(i\omega t)$, the branch cuts of the multivalued functions $\gamma_j(j = 1, 2)$ are determined as

$$\gamma_j = \begin{cases} |\sqrt{s^2-k_j^2}| & \text{if } s > k_j \\ i|\sqrt{s^2-k_j^2}| & \text{if } s < k_j \end{cases}$$

Detailed description of the radiation conditions as well as how to choose the required branch cut of the multivalued functions $\gamma_j(j = 1, 2)$ can be found in Eringen and Suhubi (1975) and Achenbach (1984).

Using (31) and the boundary condition (8_3), it can be shown that

$$\sigma_z^*(r,0) = \mu\mathscr{H}_0[N(s)V^{\mathrm{H}}(s,0);s\to r] \tag{32}$$

where

$$\begin{aligned} V^{\mathrm{H}}(s,z) &= \mathscr{H}_0[u_z^*(r,z);r\to s] \\ N(s) &= k_2^{-2}\gamma_1^{-1}[(2s^2-k_2^2)^2-4s^2\gamma_1\gamma_2] \end{aligned}$$

Now in view of the boundary conditions (8_1) and (8_2) and using eqn (32), we obtain the following dual integral equations:

$$\begin{aligned} &\int_0^\infty sN(s)V^{\mathrm{H}}(s,0)J_0(rs)\,ds = -\frac{p(r)}{\mu}, \quad 0 \leqslant r < a \\ &\int_0^\infty sV^{\mathrm{H}}(s,0)J_0(rs)\,ds = 0, \quad r > a \end{aligned} \tag{33}$$

By writing $x = r/a$, $y = as$ and $k = k_2 a$, the dual integral equations (33) can be rewritten in the following dimensionless form:

$$S_{1/2,-1}[1+H(x)]\psi(x) = f(x), 0 \leqslant x < 1$$
$$S_{0,0}\psi(x) = g(x), x > 1 \tag{34}$$

where

$$H(x) = -\left[1+\frac{M(x)}{2(1-\varepsilon^2)x}\right]$$

$$M(x) = \frac{(2x^2-k^2)^2 - 4x^2(x^2-\varepsilon^2k^2)^{1/2}(x^2-k^2)^{1/2}}{k^2(x^2-\varepsilon^2k^2)^{1/2}}$$

$$\psi(x) = a^{-1}V^{H}(x/a, 0)$$

and $S_{\eta,\alpha}$ is the Hankel operator defined by

$$S_{\eta,\alpha}f(x) = 2^{\alpha}x^{-\alpha}\int_0^{\infty} t^{1-\alpha}J_{2\eta+\alpha}(xt)f(t)\,dt$$

In eqn (34) we have $f(x) = f_1(x) = xa^2p(xa)/4\mu(1-\varepsilon^2)$ for $0 \leqslant x < 1$ and $f(x) = f_2(x)$ is unknown for $x > 1$, while $g(x) = g_1(x)$ is unknown for $0 \leqslant x < 1$ and $g(x) = g_2(x) = 0$ for $x > 1$.

Following Cooke (1965), we put Sneddon's trial solution $\psi = S_{0,1/2}q$ into eqn (34) to reduce them to the following Fredholm integral equation of the second kind via the auxiliary function $q(x)$:

$$q(x) + \int_0^1 K(x,u)q(u)\,du = r(x) \tag{35}$$

where

$$K(x,u) = \frac{2k}{\pi}\int_0^{\infty} H(kt)\sin(xtk)\sin(ukt)\,dt$$

$$r(x) = \frac{a^2}{2\mu\pi^{1/2}(1-\varepsilon^2)}\int_0^x \frac{up(ua)\,du}{(x^2-u^2)^{1/2}}. \tag{36}$$

The integral equation (35) is basically the same as that obtained by Sih and Loeber (1969) and Robertson (1967).

Applying contour integration to the integral (36_1) [see Robertson (1967)]; the kernel $K(x,u)$ can be reduced to the following form suitable for numerical computation:

$$K(x,u) = \frac{-4k}{\pi(1-\varepsilon^2)}\left[\int_0^{\varepsilon} \frac{(s^2-1/2)^{1/2}}{s(\varepsilon^2-s^2)^{1/2}}\sin(xsk)\exp(-iusk)\,ds\right.$$
$$\left. + \int_0^1 s(1-s^2)^{1/2}\sin(xsk)\exp(-iusk)\,ds\right], \quad x < u. \tag{37}$$

The expression for the kernel $K(x,u)$ for the case where $x > u$ can be obtained by simply interchanging the positions of x and u in eqn (37).

5. DYNAMIC STRESS INTENSITY FACTOR

It is well known that in case of time-harmonic vibration, stress intensity factor is determined as

$$K_I(t) = K_I^* \exp(i\omega t)$$

Here K_I^* is the complex amplitude of the SIF, which is determined as:

$$K_I^* = \lim_{r \to a+0} [2(r-a)]^{1/2} \sigma_z^{*(2)}(r, 0) = \frac{4\mu(1-\varepsilon^2)}{a^{3/2}[2\pi(r-a)]^{1/2}} q(1)$$

which can be represented as $K_I^* = \alpha K_{st} \exp(-i\delta)$ so that

$$\alpha = \frac{4\mu(1-\varepsilon^2)}{K_{st} a(\pi a)^{1/2}} |q(1)|$$

$$\delta = -\arctan \frac{\mathscr{T} K_I^*}{\mathscr{R} K_I^*} \tag{38}$$

where α is some coefficient which depends on the properties of the elastic material and mutual position of the crack and the loads. K_{st} is the *static SIF*, for the case where the corresponding loads are applied *directly* on the crack surfaces.

We next introduce the notation

$$h(x) = \frac{4\mu(1-\varepsilon^2)}{a(\pi a)^{1/2} K_{st}} q(x) \tag{39}$$

With this notation, from eqn (38) we have

$$\alpha = |h(1)|$$

$$\delta = -\arctan \frac{\mathscr{T} h(1)}{\mathscr{R} h(1)} \tag{40}$$

In view of the notation in eqn (39), the governing integral equation (35) is reduced to

$$h(x) + \int_0^1 K(x, u) h(u) \, du = \frac{2a^{1/2}}{\pi K_{st}} \int_0^x \frac{u p(ua) \, du}{(x^2 - u^2)^{1/2}} \tag{41}$$

By making the change of variable $u = x \sin \theta$ in the right-hand side of eqn (41), we rewrite eqn (41) in the following form more suitable for numerical computation:

$$h(x) + \int_0^1 K(x, u) h(u) \, du = \frac{2xa^{1/2}}{\pi K_{st}} \int_0^{\pi/2} \sin \theta p(xa \sin \theta) \, d\theta \tag{42}$$

6. NUMERICAL RESULTS AND DISCUSSION

The integral equation (42) has been solved numerically for two body force loading cases. All computations in this paper have been carried out for Poisson's ratio, $\nu = 0.25$.

Case 1

The elastic space with the penny-shaped crack is stretched by uniform axial loads of constant intensity σ, acting over a circular area of radius a (same as the radius of the crack) and placed symmetrically with respect to the crack plane $z = 0$ at some distance z_1 from it. For this case the functions $F_z^*(r, z)$ and $p(r)$ are given by eqns (29) and (30), respectively, with c replaced by a. Note that this case reduces to that considered by Robertson (1967) and Sih and Loeber (1969) when the body force placement distance tends to zero. The static SIF corresponding to the case where uniform axial pressures are placed *directly* on the crack faces is given by Kassir and Sih (1975) and Andreikiv (1982):

$$K_{st} = \frac{2\sigma a^{1/2}}{\pi}. \tag{43}$$

Putting eqn (43) into eqn (42), we obtain the integral equation to be solved for this case:

$$h(x) + \int_0^1 K(x,u)h(u)\,du = \frac{x}{\sigma}\int_0^{\pi/2} \sin\theta p(xa\sin\theta)\,d\theta. \tag{44}$$

The corresponding static solution can be obtained by letting ω tend to zero in eqn (44) in which case the kernel $K(x,u)$ tends to zero and we obtain as per eqn (40)

$$\alpha_{st} = h(1) = \frac{1}{\sigma}\int_0^{\pi/2} \sin\theta p_{st}(a\sin\theta)\,d\theta \tag{45}$$

where $p_{st}(r)$ is the static counterpart of the solution (30) which can be obtained by letting ω tend to zero, namely

$$p_{st}(r) = \frac{\sigma z_1}{2\pi(1-\nu)}\int_0^a r_0\,dr_0 \int_0^{\pi}\left(\frac{1-2\nu}{R_1^3} + 3\frac{z_1^2}{R_1^5}\right)d\phi \tag{46}$$

where $R_1 = (r^2 + r_0^2 + z_1^2 - 2rr_0\cos\phi)^{1/2}$.

We have first determined the variations of the normalized static SIF, α_{st}, with the normalised body force placement distance, $H = z_1/a$ by using numerical integration rule to the integrals in eqns (45) and (46), namely, integrals with respect to θ and ϕ have been evaluated using 32-point Gaussian quadrature rule while the integral with respect to r_0 has been evaluated by using 8-point Gaussian integration of moments (Abramowitz and Stegun, 1965). The results thus obtained are illustrated graphically by the solid line in Fig. 1. We have next solved the integral eqn (44) using 32-point Gaussian quadrature rule to determine the normalized dynamic SIF, α, for different values of the longitudinal wave number, k_1a and the normalized body force placement distance $H = z_1/a$. These results are represented in Figs 1 and 2. As can be seen from the Fig. 1, dynamic SIF generally decreases with H. Note however that at bigger values of the parameters k_1a and H, the dynamic stress intensity factor oscillates. Figure 2 illustrates the variations of α with the longitudinal wave number for different values of the normalized body force placement distance. One can observe the dynamic SIF overshoots, i.e. the peaks of the values of the dynamic stress intensity factor with respect to the static one, for certain ranges of frequency. This means, of course, that the possibility of an abrupt catastrophic crack propagation increases in the dynamic case, in comparison with the analogous static situation under the same external loading amplitude. For $H = 0$, the first peak value is approximately 1.478 (for $\nu = 0.25$) which occurs at $k_1a = 0.85$. This result has been obtained by Sih and Loeber (1969) and confirmed by our method. At low frequency ranges, the results obtained are found to be in good agreement with the corresponding static result, being represented by the solid line in Fig. 1.

Case 2

As the second example, we consider the case where the elastic space with the penny-shaped crack is acted upon by two concentrated loads P distributed uniformly over a circular region of radius $b(b \leqslant a)$ and placed symmetrically with respect to the crack plane, $z = 0$. For this case

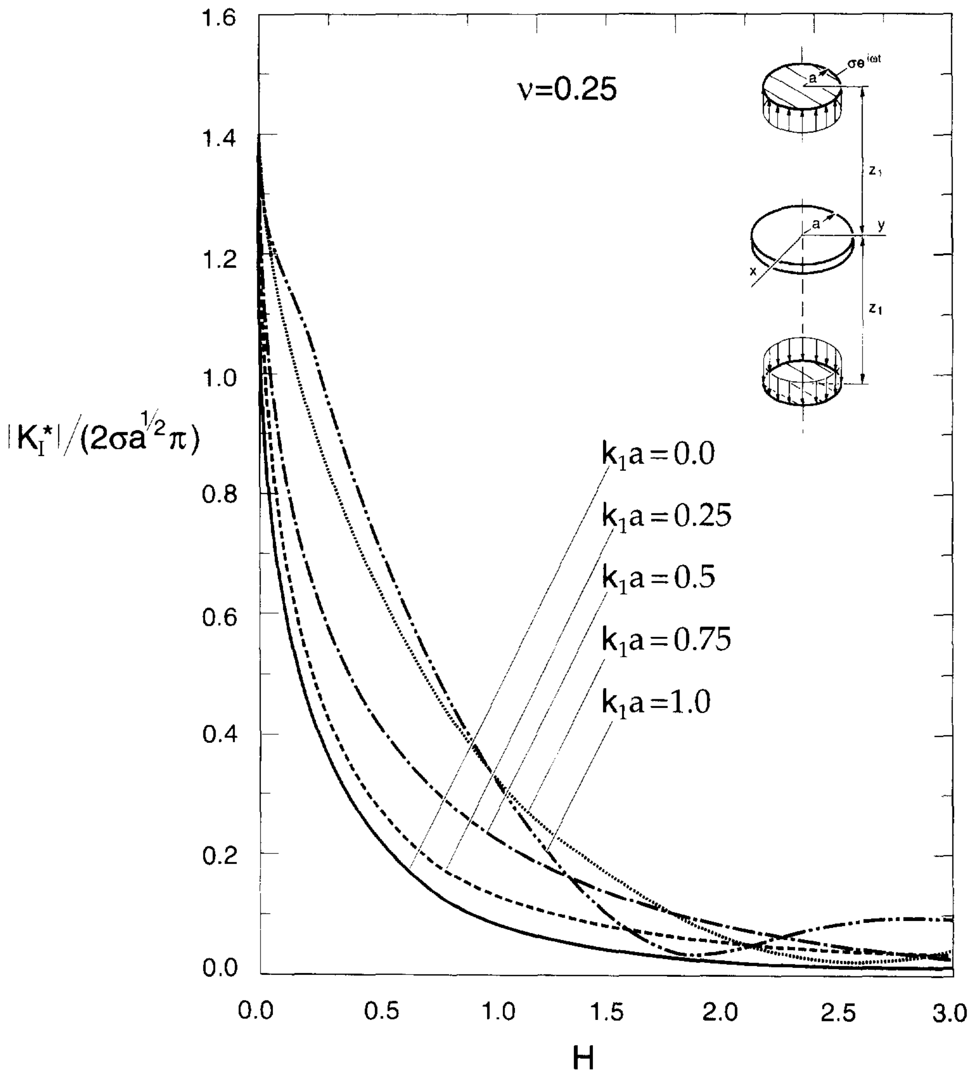

Fig. 1. Variations of the normalized dynamic SIF with the normalized body force placement distance for different values of the longitudinal wave number.

$$F_z^*(r,z) = \frac{P}{\pi b^2}H(b-r)[\delta(z-z_1)-\delta(z+z_1)]. \tag{47}$$

Putting this expression into eqn (25), we obtain

$$p(r) = \sigma_z^{*(1)}(r,0) = \frac{4P}{b^2}\int_0^b r_0 G_\sigma^*(r,r_0\,;0,z_1)\,\mathrm{d}r_0. \tag{48}$$

The static stress intensity factor corresponding to the case where tractions of the form (47) act directly on the crack borders is given by Kassir and Sih (1975) and Andreikiv (1982):

$$K_{st} = \frac{2Pa^{1/2}}{\pi^2 b^2}\left(1-\sqrt{1-\frac{b^2}{a^2}}\right). \tag{49}$$

We shall use eqn (49) as the normalizing SIF for this case.

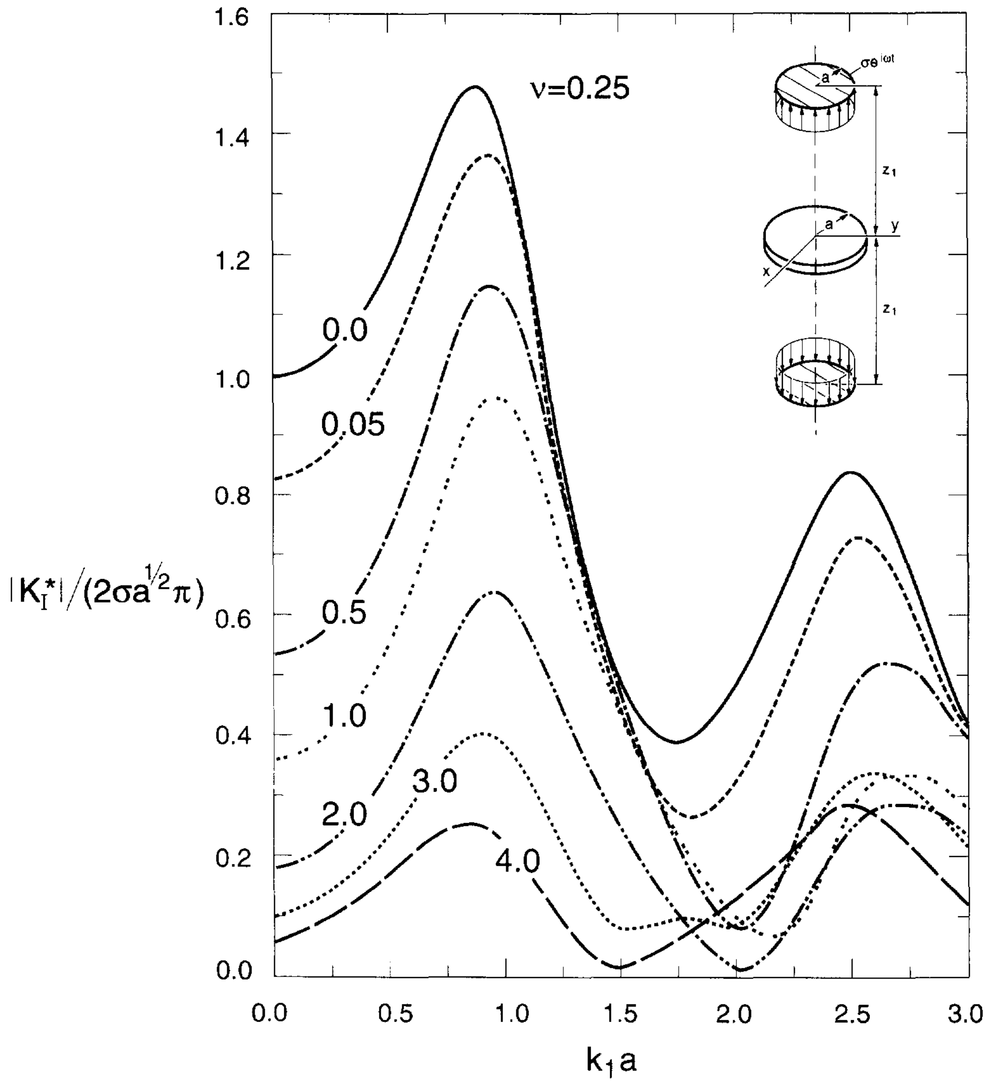

Fig. 2. Variations of the normalized dynamic SIF with the longitudinal wave number for different values of the normalized body force placement distance.

Putting the expression (49) into eqn (42), we get the governing integral equation for Case 2:

$$h(x)+\int_0^1 K(x,u)h(u)\,\mathrm{d}u = \frac{\pi b^2 x}{P\left(1-\sqrt{1-\dfrac{b^2}{a^2}}\right)}\int_0^{\pi/2}\sin\theta p(xa\sin\theta)\,\mathrm{d}\theta. \tag{50}$$

Note that this case reduces to the Case 1 if $a = b$. Also, note that at $b \to 0$, we have the case where the elastic space is subjected to the action of two symmetrically placed time-harmonic concentrated loads, for which case the corresponding function $p(r)$ is given by eqn (27).

The integral equation corresponding to the case where the crack faces are *directly* acted upon by the stresses of the form (47) (that is, $H = 0$) can easily be shown to be

$$h(x)+\int_0^1 K(x,u)h(u)\,\mathrm{d}u = s(x) \tag{51}$$

where

$$s(x) = \frac{x}{1-\sqrt{1-\frac{b^2}{a^2}}}, \quad x \leqslant \frac{b}{a} < 1$$

$$s(x) = \frac{x\left(1-\sqrt{1-\frac{b^2}{x^2a^2}}\right)}{1-\sqrt{1-\frac{b^2}{a^2}}}, \quad x > \frac{b}{a}. \tag{52}$$

The static solution corresponding to Case 2 can be obtained by letting ω tend to zero in eqn (50) in which case $K(x, u)$ tends to zero and we find that

$$\alpha_{st} = \frac{\pi b^2}{P\left(1-\sqrt{1-\frac{b^2}{a^2}}\right)} \int_0^{\pi/2} \sin\theta p_{st}(a\sin\theta)\,d\theta \tag{53}$$

where $p_{st}(r)$ is the static counterpart of the solution (48), namely

$$p_{st}(r) = \frac{Pz_1}{2\pi b^2(1-v)} \int_0^b r_0\,dr_0 \int_0^\pi \left(\frac{1-2v}{R_1^3} + 3\frac{z_1^2}{R_1^5}\right) d\phi \tag{54}$$

where $R_1 = (r^2+r_0^2+z_1^2-2rr_0\cos\phi)^{1/2}$.

Closed-form expression for α_{st} given by eqns 53 and 54 can be obtained only for the case where $b = 0$, which corresponds to the case where the elastic space with the penny-shaped crack is under the action of two concentrated loads placed symmetrically with respect to the crack plane. Indeed, by letting b tend to zero in eqns (53) and (54), we obtain:

$$\alpha_{st} = \frac{H}{2(1-v)} \int_0^{\pi/2} \left(\frac{1-2v}{R_H^3} + 3\frac{z_1^2}{R_H^5}\right) \sin\theta\,d\theta \tag{55}$$

where $R_H = (\sin^2\theta + H^2)^{1/2}$, $H = z_1/a$.

The integrals in eqn (55) are evaluated in closed forms using the integral 3.676.1 from Gradshteyn and Ryzhik (1980) and is as follows:

$$\alpha_{st} = \frac{1+\Lambda H^2}{(1+H^2)^2} \tag{56}$$

where $\Lambda = (2-v)/(1-v)$, which is the well-known classical solution by Sneddon and Tweed (1967a).

It can be easily seen from eqn (56) that the static SIF in this case attains the maximum value of $(2-v)^2/4(1-v)$ at $H = \sqrt{v/(2-v)}$.

We now summarize the steps of the numerical computations carried out for this case:

1. Using eqns (53), (54) and (56), we have evaluated the normalized static SIF for different values of the normalized body force placement distance, $H = z_1/a$, and the parameter b/a. As before the integrals involved in these expressions have been computed by using 32-point Gaussian quadrature rule and 8-point Gaussian integration of moments.

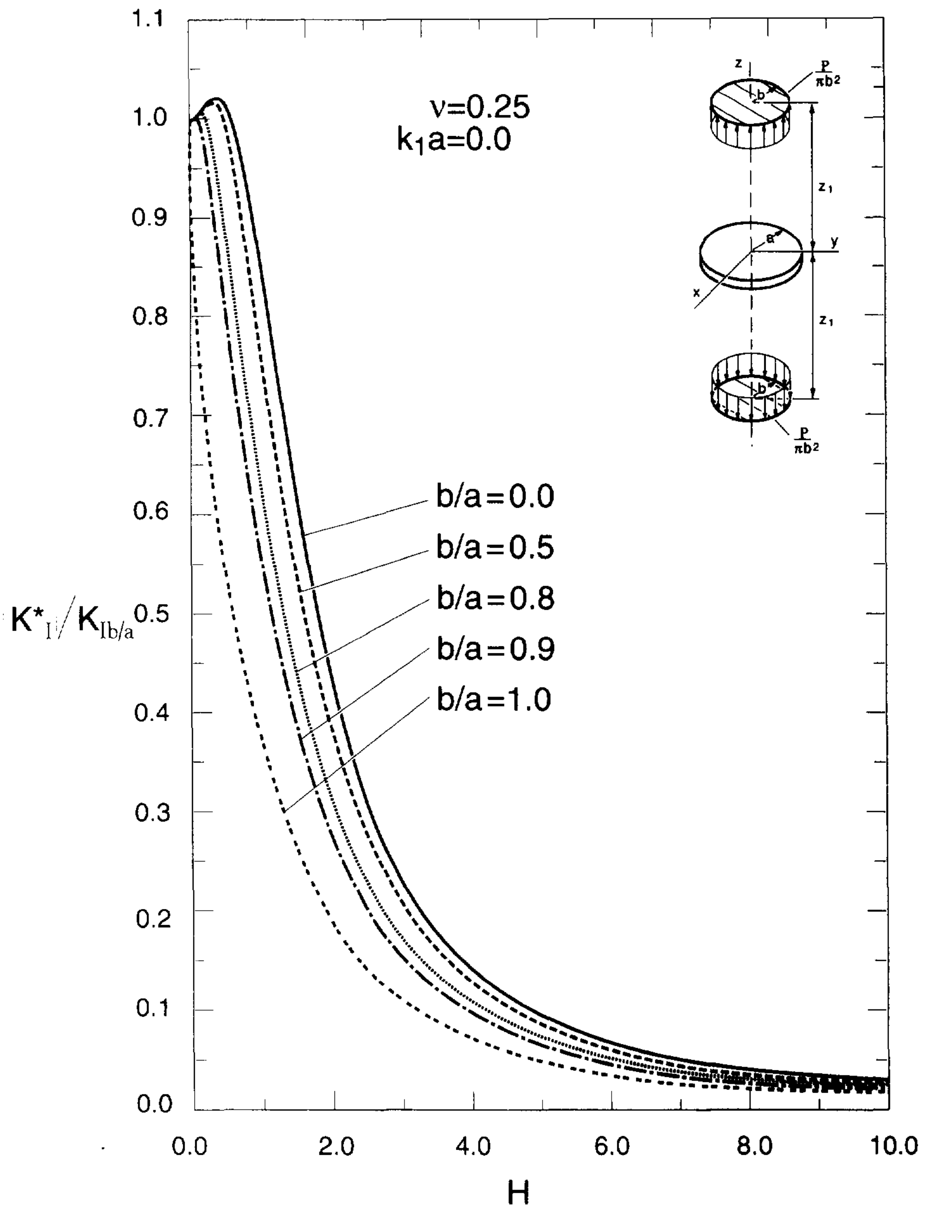

Fig. 3. Variations of the normalized static SIF with the normalized body force placement distance for different values of b/a.

These results are shown in Fig. 3, where $K_{Ib/a}$ denotes the normalizing stress intensity factor given by eqn (49). It is interesting to note that within a wide range of the values of the parameter H, stress intensity factor changes very little. This observation may be useful in experimental investigation concerning the determination of fracture toughness of materials.

2. We have next solved the integral equations (50) and (51) also using 32-point Gaussian quadrature rule for different values of the parameters k_1a, H and b/a. These results are illustrated graphically in Figs 4 to 13, where, as before, $K_{Ib/a}$ denotes the normalizing stress intensity factor given by eqn (49). Though numerical computations have been carried out for a wide range of values of the parameter b/a, figures have been plotted for $b/a = 1.0, 0.9, 0.8, 0.7, 0.5$ in order that the plots not be overcrowded. From these plots, it is clear that the stress intensity factor at the vicinity of the penny-shaped crack increases, *ceteris paribus*, with the decrease of the parameter b/a, which is quite obvious. On the otherhand, SIF usually decreases with H, except for a range of the values of H between 0.0

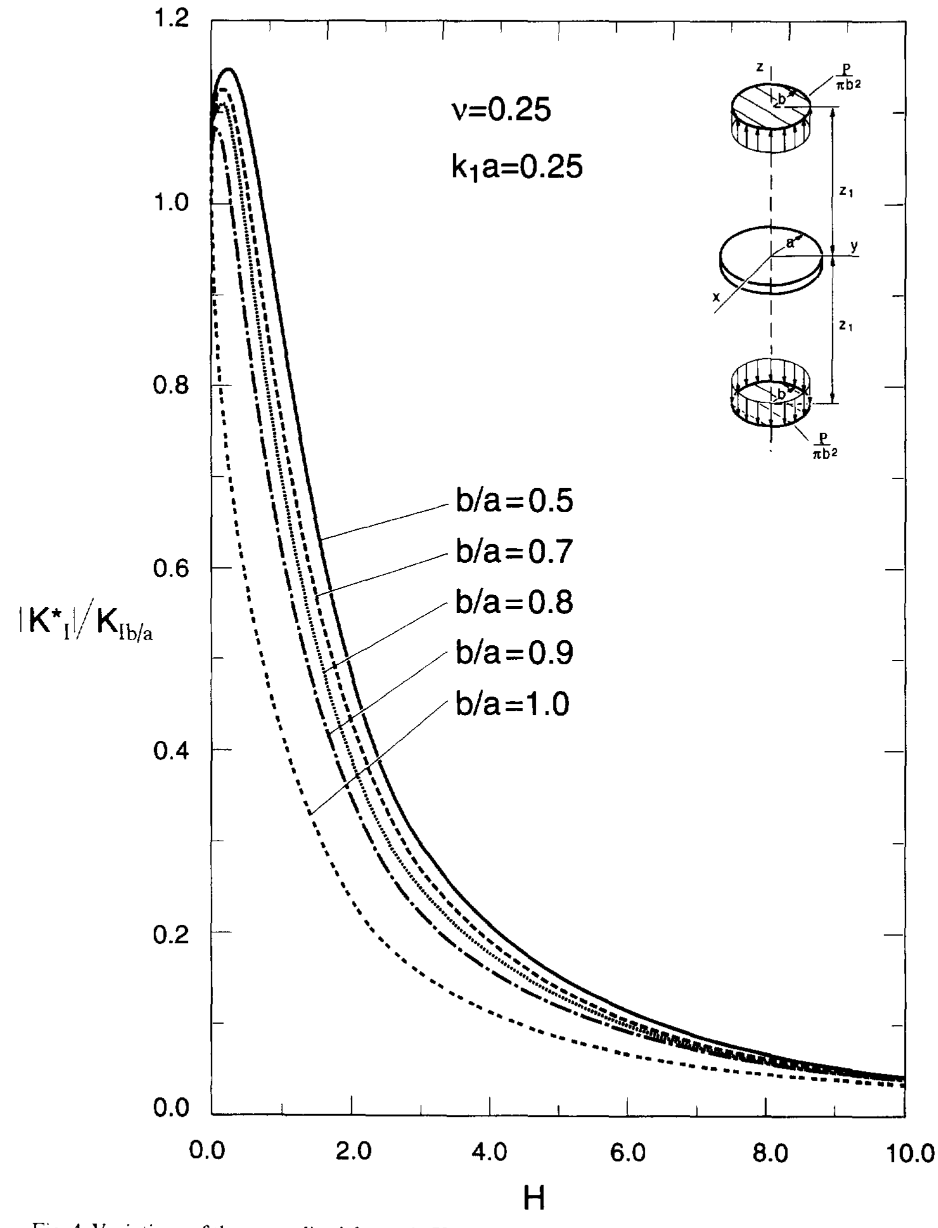

Fig. 4. Variations of the normalized dynamic SIF with the normalized body force placement distance for different values of b/a ($k_1a = 0.25$).

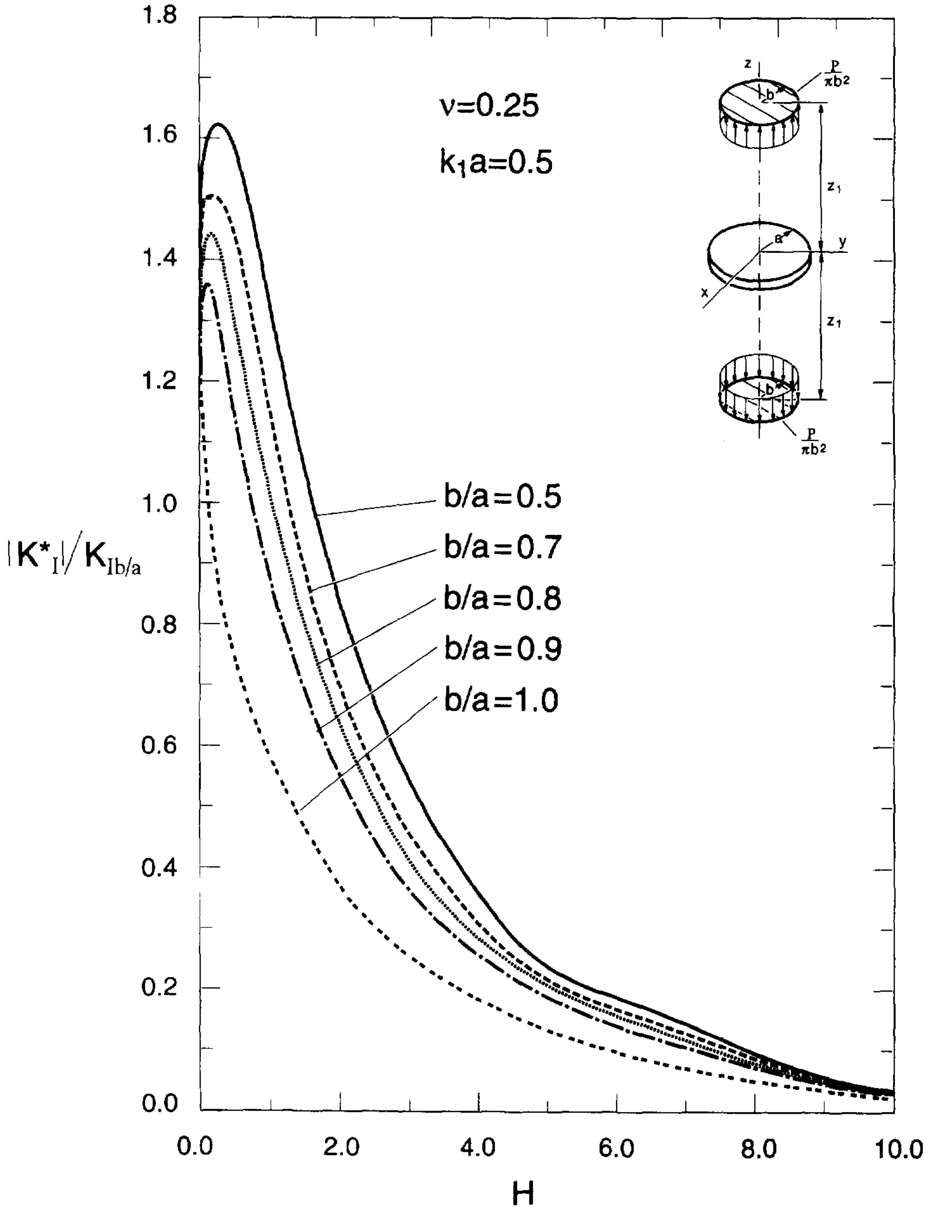

Fig. 5. Variations of the normalized dynamic SIF with the normalized body force placement distance for different values of b/a ($k_1a = 0.5$).

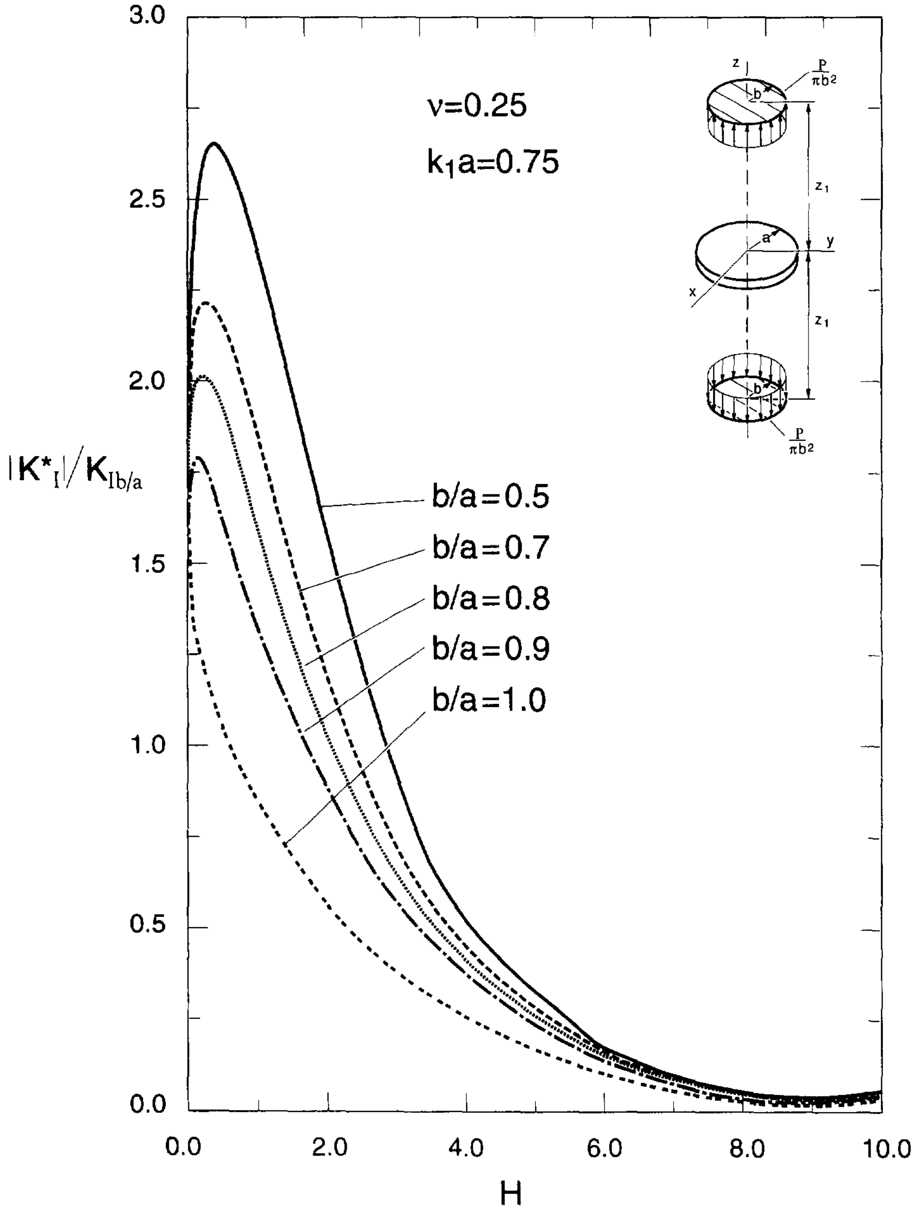

Fig. 6. Variations of the normalized dynamic SIF with the normalized body force placement distance for different values of b/a ($k_1a = 0.75$).

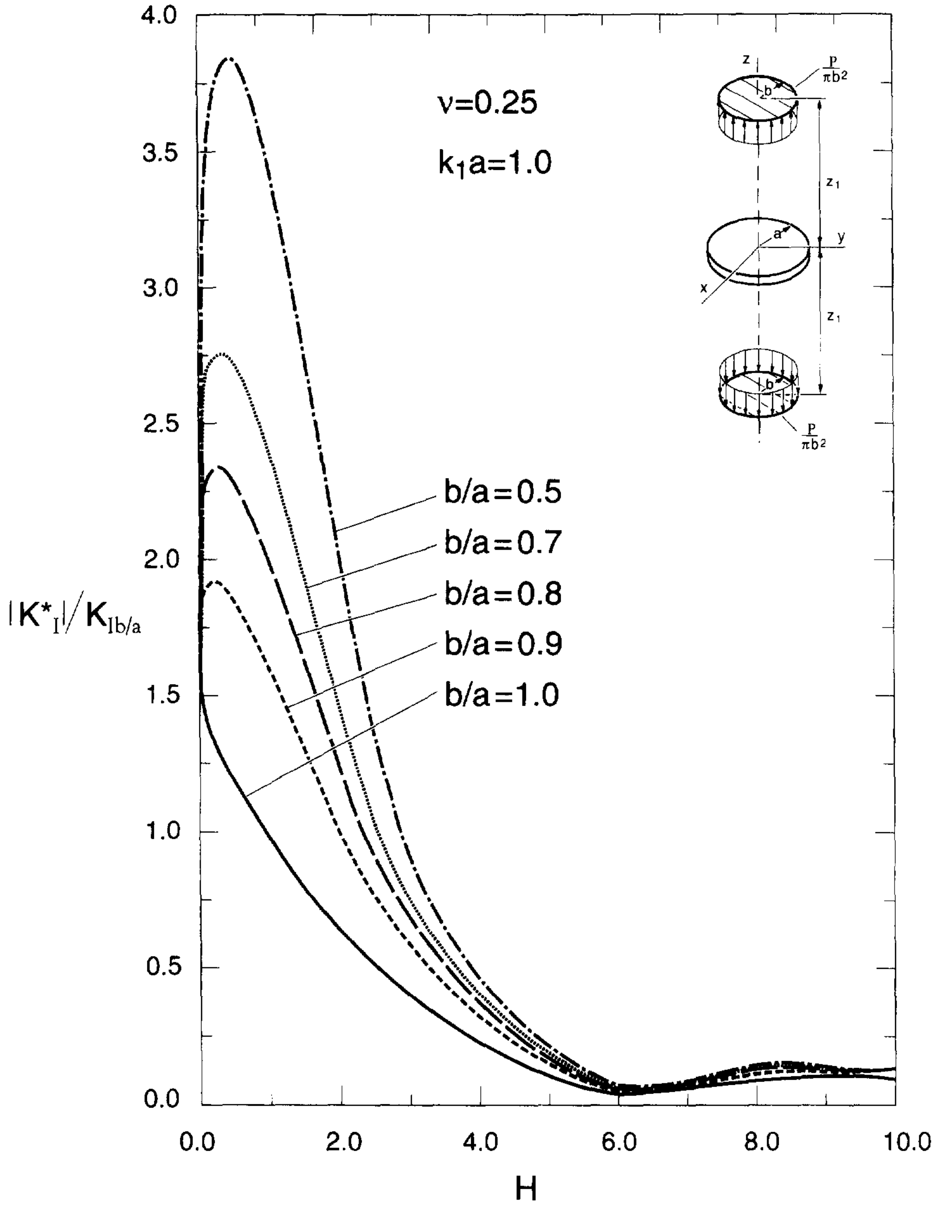

Fig. 7. Variations of the normalized dynamic SIF with the normalized body force placement distance for different values of b/a ($k_1a = 1.0$).

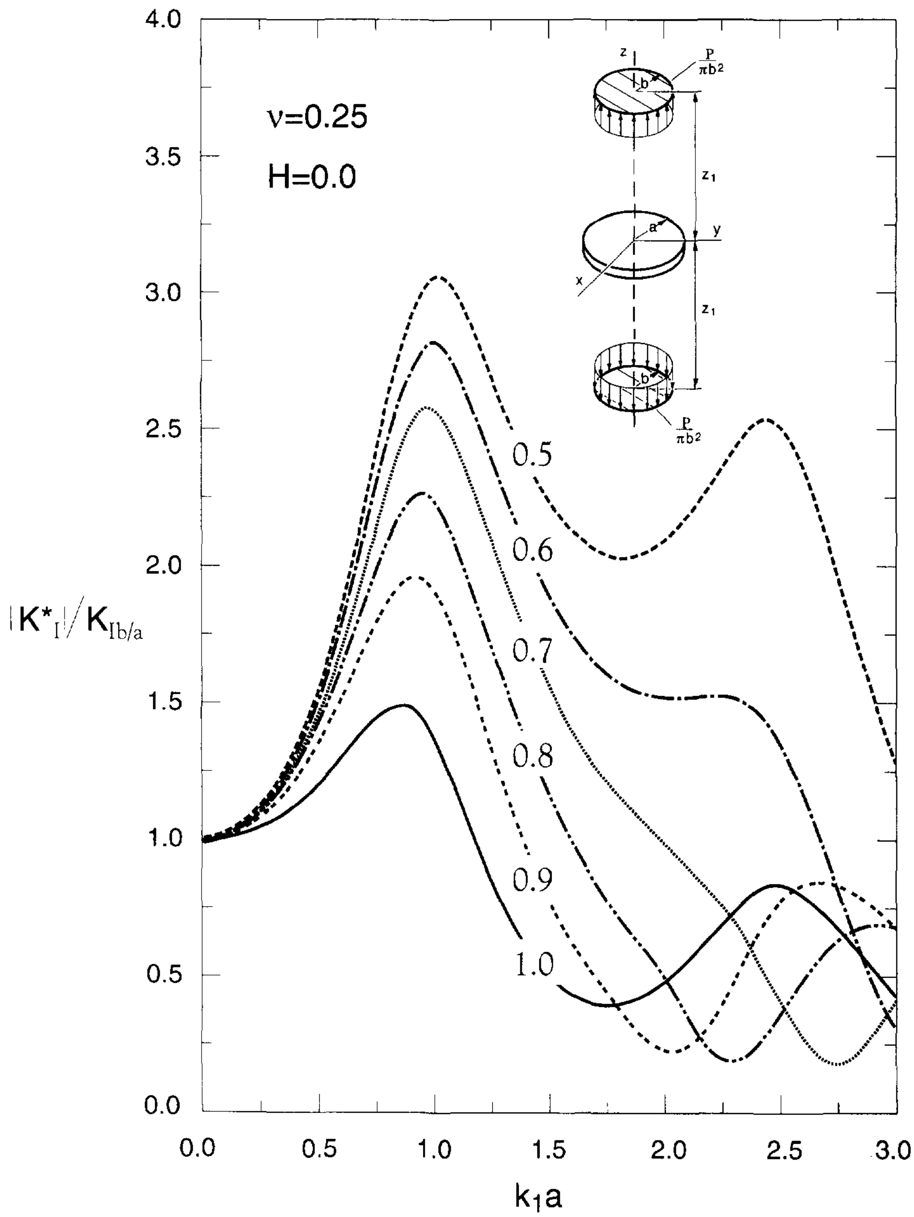

Fig. 8. Variations of the normalized dynamic SIF with the longitudinal wave number for different values of b/a ($H = 0$).

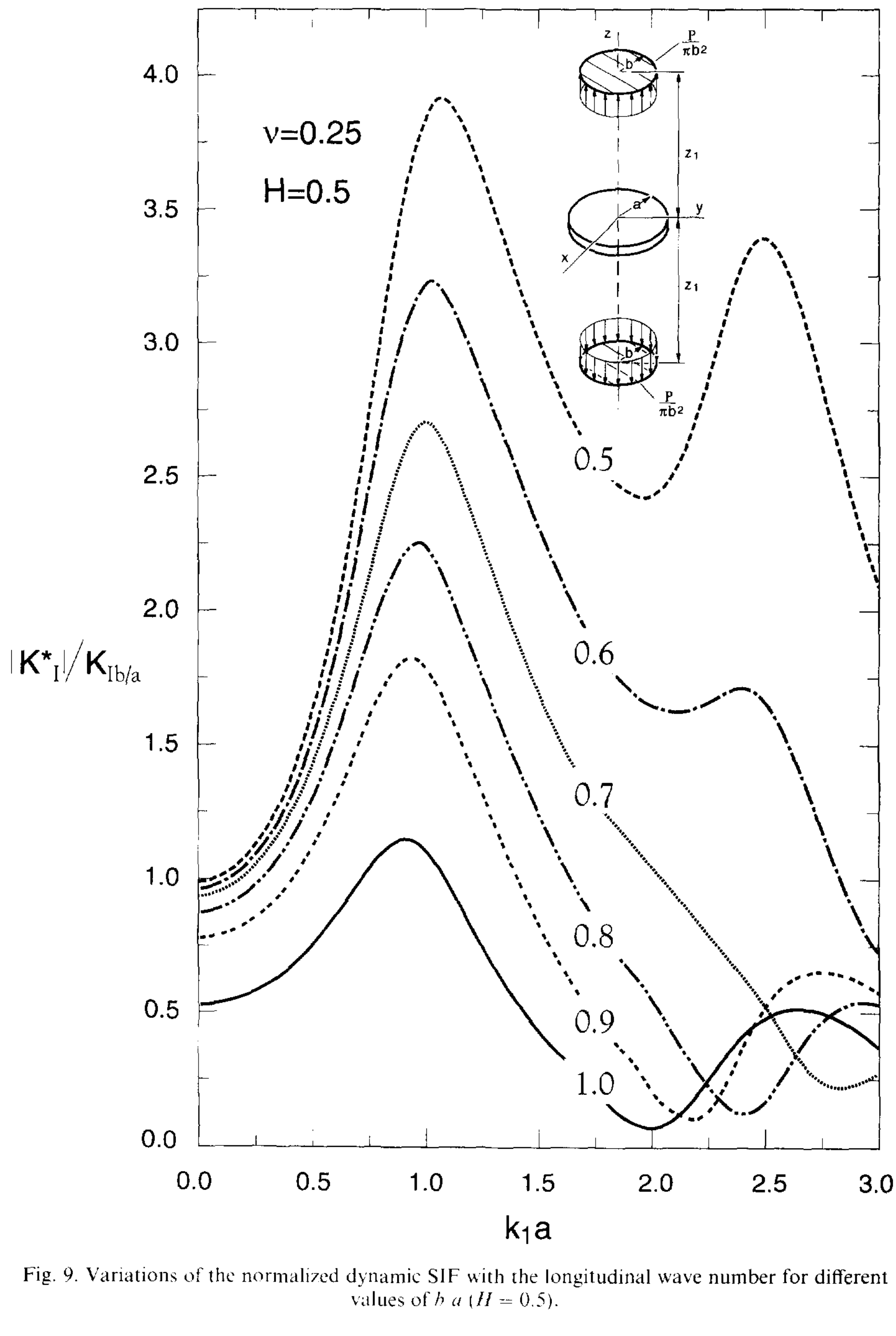

Fig. 9. Variations of the normalized dynamic SIF with the longitudinal wave number for different values of b/a ($H = 0.5$).

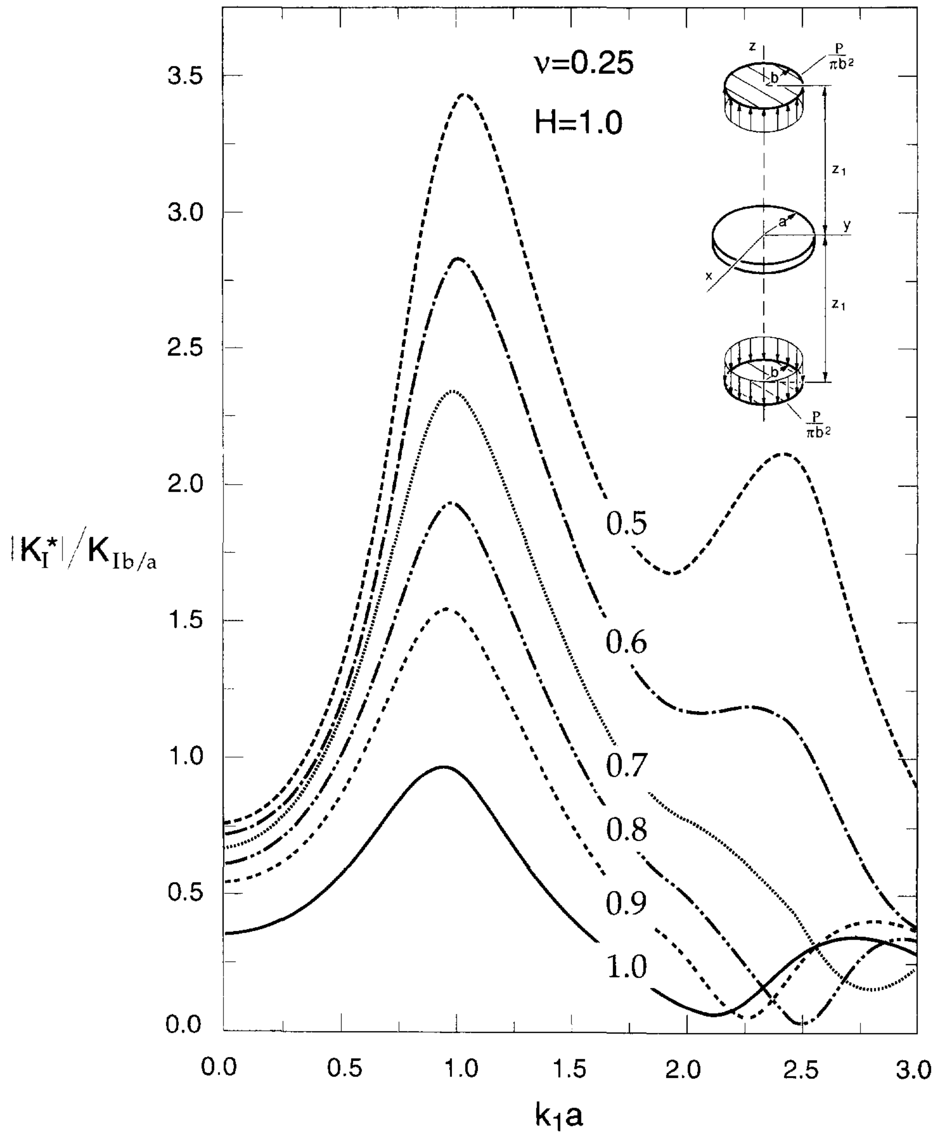

Fig. 10. Variations of the normalized dynamic SIF with the longitudinal wave number for different values of b/a ($H = 1.0$).

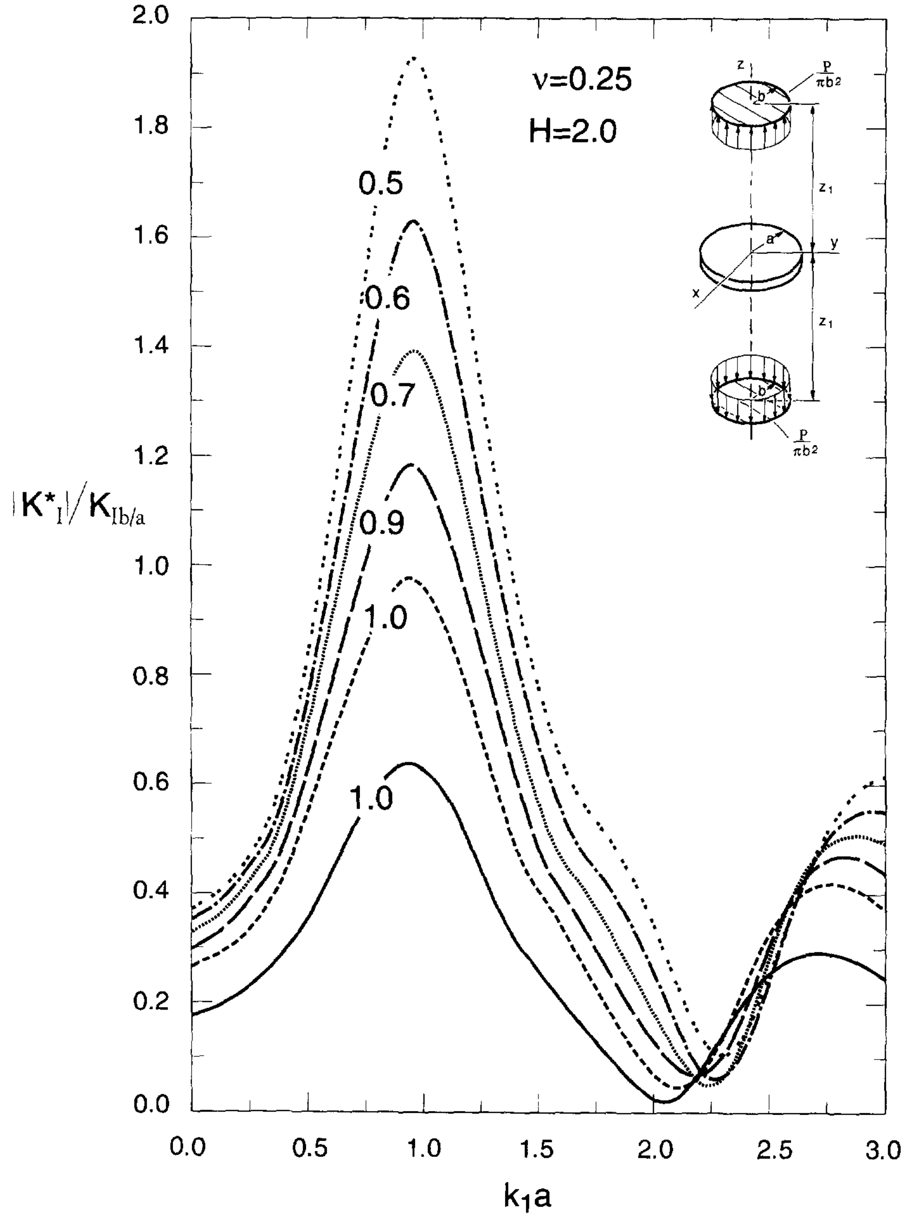

Fig. 11. Variations of the normalized dynamic SIF with the longitudinal wave number for different values of b/a ($H = 2.0$).

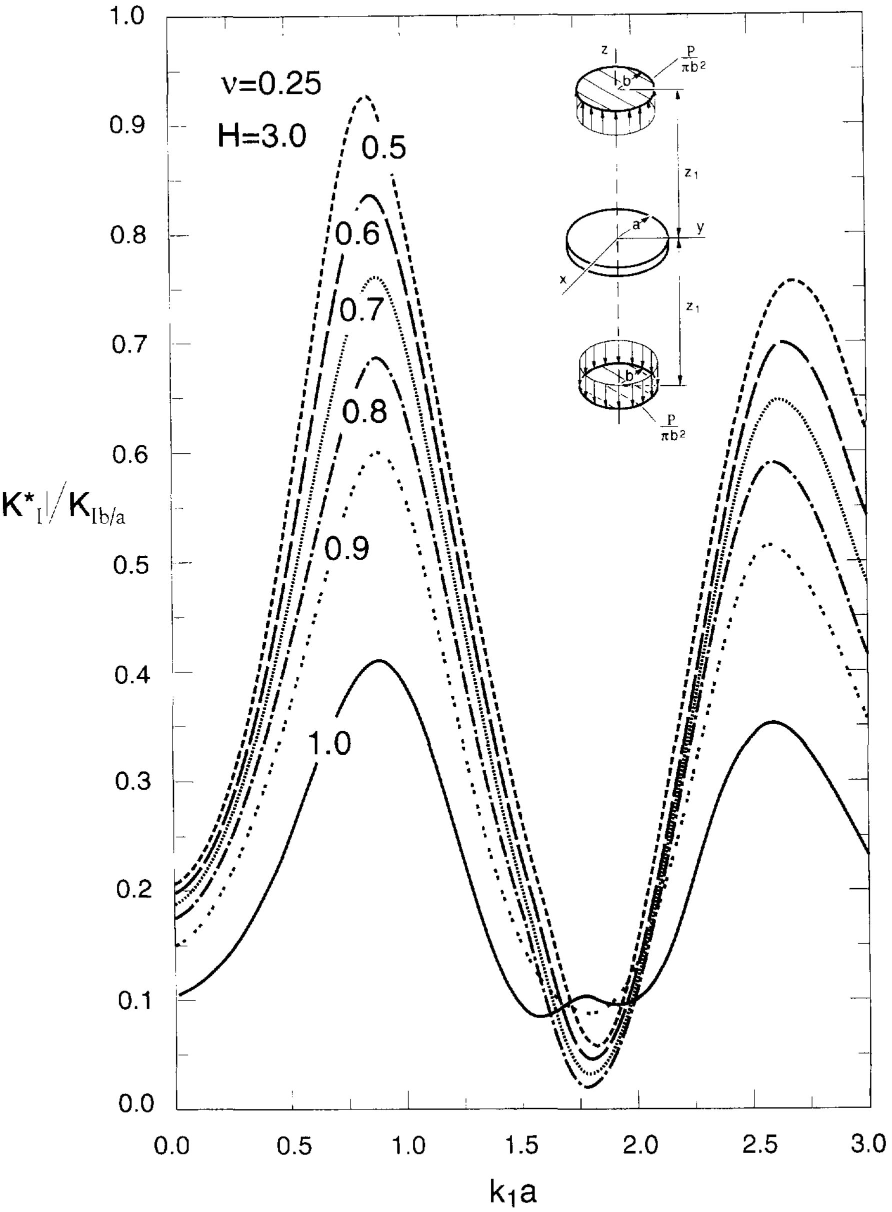

Fig. 12. Variations of the normalized dynamic SIF with the longitudinal wave number for different values of b/a ($H = 3.0$).

and 0.5, where the stress intensity factor, similar to the corresponding static cases represented by Fig. 3, increases in comparison with those values obtained for the cases where the loads are placed directly on the crack surfaces. In numerical computations, 2, 4, 6 and 15% increases of the stress intensity factor (with respect to the value of the SIF corresponding to the case where the loads are placed *directly* on the crack faces) for $k_1 a = 0.75$ have been observed for $b/a = 0.9$, 0.8, 0.7 and 0.5, respectively for $0 < H < 0.5$.

7. CONCLUDING REMARKS

In the present paper we have investigated the axisymmetric problem of determining the dynamic SIF at the rim of a penny-shaped crack embedded in an elastic space in which time-harmonic axial body forces are available. The solution of the problem has been

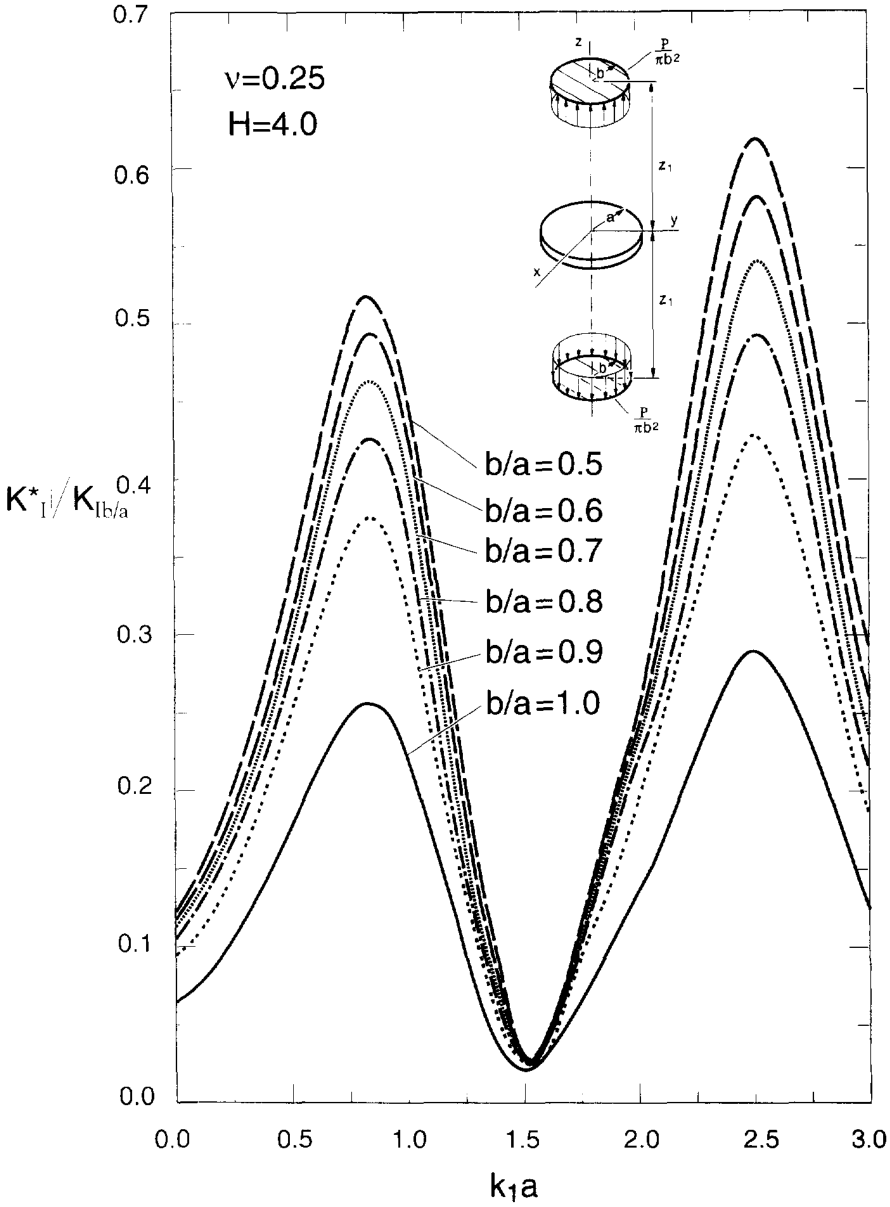

Fig. 13. Variations of the normalized dynamic SIF with the longitudinal wave number for different values of b/a ($H = 4.0$).

obtained by means of superposition of the solutions of two simpler problems, namely the infinite elastic space under the prescribed body forces and the infinite elastic space containing the penny-shaped crack whose faces are subjected to the action of some time-harmonic axial stresses. Fourier and Hankel transforms have been employed to solve the first problem. The method can be used, *mutatis mutandis*, for any *arbitrary axisymmetric* time-dependent body force loading cases (Rahman, 1995). The second problem has been reduced to a Fredholm integral equation of the second kind via an auxiliary function, which has been solved numerically in order to determine the variations of the dynamic SIF at the vicinity of the penny-shaped crack for some particular cases of body force loadings. Of further interest is the problem of determining the stress intensity factor for a mode-I penny-shaped crack under time-harmonic *asymmetric* body forces. Research in this direction will be reported elsewhere.

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